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An iterative method for the conformal mapping of doubly connected regions

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Abstract: An iterative method is described for the determination of the conformal mapping of a circular annulus $R_q := \{z: q < |z| < 1\}$ onto a doubly connected region G with smooth boundary curves. The method is analogous to the method of [7] for simply connected regions. It requires in each step the solution of a Riemann–Hilbert problem on R_q. This problem can be solved explicitly in terms of a generalized conjugation operator K. A degeneracy of this problem leads in a natural way to an adjustment of the parameter q in each step. If the boundary curves of G admit parametrizations by functions η_ν with second derivatives which are Hölder continuous with exponent μ, $0 < \mu \leqslant 1$, then the method converges locally in the Sobolev space W of 2π-periodic absolutely continuous functions with square-integrable derivatives. The order of convergence is at least $1 + \mu$. The convergence is quadratic if the η_ν have Lipschitz-continuous second derivatives. For the numerical implementation of the method the operator K can be approximated by a Wittich type method, which can be performed very effectively using FFT. A calculation with $N = 2^m$ grid points on the computer requires storage of the order $O(N)$ and computing time $O(mN)$. The results of some test calculations are reported and compared with results of calculations with the method of Theodorsen–Garrick.

Keywords: Numerical conformal mapping, Riemann–Hilbert problem.

1. Introduction

Let G be a doubly connected region in the complex plane with boundary consisting of two closed Jordan curves Γ_1 and Γ_2. Then there exists a number q in the interval $0 < q < 1$ and a function Φ analytic in the circular annulus $R_q := \{q < |z| < 1\}$ such that Φ is a conformal mapping of R_q onto the region G. The reciprocal $M := 1/q$ is called the module of G.

By the Carathéodory–Osgood theorem, Φ can be extended continuously to the boundary of R_q. For $\nu = 1, 2$, let Γ_ν be parametrized by a 2π-periodic complex function $\eta_\nu(s)$ in such a way that η_ν runs through Γ_ν in the counter-clockwise direction. We assume that Γ_1 is the outer boundary of G. Then the boundary values of Φ can be represented in the form

$$\Phi(e^{it}) = \eta_1(S_1(t)), \qquad \Phi(q\, e^{it}) = \eta_2(S_2(t)) \tag{1.1}$$

with real functions S_ν such that $S_\nu(t) - t$ is 2π-periodic. The conformal mapping is uniquely determined if the image of one boundary point is prescribed. We choose here the normalization

$$\Phi(1) = \eta_1(0). \tag{1.2}$$

Since the whole function Φ can easily be reconstructed from its boundary values (1.1) by means of Cauchy's formula, it is sufficient to determine the two functions S_1 and S_2 and the number q. There are several numerical methods for the calculation of Φ. (See [3, Chapter V]. A survey of more recent methods is given in [2].) Many of these methods are modelled after methods for simply connected regions. Doubly connected regions are more difficult, since the canonical region R_q is not known in advance. The number q must be calculated.

We carry over our iterative method for the conformal mapping of simply connected regions as discussed in [7,8] to the doubly connected case. To this end we discuss first the Riemann–Hilbert problem on a circular annulus. It turns out that the solution of this problem can be represented in closed form in terms of a conjugation operator. In each step our iterative method calculates from approximations q_k, $S_{\nu,k}$ for q and S_ν new approximations q_{k+1} and $S_{\nu,k+1}$ by solving a certain Riemann–Hilbert problem on an annulus R_p. It turns out that for fixed p this problem does not in general admit a solution subject to the normalization (1.2). But one can construct an approximate solution which satisfies the Riemann–Hilbert problem with a certain additional term added. The structure of this error term shows clearly that it is due to an inadequate choice of p. This observation allows one to determine q_{k+1} in a very natural way. If the η_ν have Lipschitz continuous derivatives $\dot{\eta}_\nu$, then the iteration can be continued indefinitely as long as q_k remains in the interval $0 < q_k < 1$. If η_ν have Hölder continuous second derivatives, the method converges locally in the Sobolev space W of 2π-periodic absolutely continuous functions with derivatives in L^2. A discretization is described which requires in each iterative step 12 FFTs of length N, where N is the number of grid points. We report on some test calculations and finally compare our results with results calculated by the method of Theodorsen–Garrick.

2. A family of operators

We discuss here a two-parameters family of operators which are useful for solving boundary value problems for analytic functions on an annulus. Let L^2 be the space of complex 2π-periodic functions which are square-integrable over the interval $[0, 2\pi]$. For functions ϕ in L^2 the norm is defined by

$$\|\phi\|_2 := \left(\int_0^{2\pi} |\phi(t)|^2 \mathrm{d}t\right)^{1/2}.$$

We denote by W the Sobolev space of complex 2π-periodic absolutely continuous functions ϕ with derivative ϕ' in L^2. For functions ϕ in W the norm is defined in terms of the maximum-norm $\|\cdot\|_\infty$ and the L^2-norm $\|\cdot\|_2$ by

$$\|\phi\|_W := \max(\|\phi\|_\infty, \|\phi'\|_2).$$

With these norms L^2 and W are Banach spaces with $\mathbb{C}$ as coefficient field. Sometimes it will be necessary to consider spaces of real functions. We denote by L^2_r and W_r the sets of real functions ϕ in L^2 and W, respectively. Provided with the norms $\|\cdot\|_2$ and $\|\cdot\|_W$, these subsets are Banach spaces with $\mathbb{R}$ as coefficient field.

For real numbers p, λ with $0 < p < 1$, the operator $K_{p,\lambda}$ is defined on $L^2_\mathrm{r} \times L^2_\mathrm{r}$. If ϕ_1, ϕ_2 are real functions in L^2 with Fourier series

$$\phi_\nu(t) \sim a_{\nu,0} + \sum_{n=1}^{\infty} (a_{\nu,n}\cos nt + b_{\nu,n}\sin nt) \tag{2.1}$$

for $\nu = 1, 2$, then $K_{p,\lambda}(\phi_1, \phi_2)$ is represented by a Fourier series

$$K_{p,\lambda}(\phi_1, \phi_2)(t) \sim \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt), \tag{2.2}$$

with real coefficients α_n, β_n which are calculated from the coefficients in (2.1) by the formula

$$\alpha_n - \mathrm{i}\beta_n = F_1(\mathrm{e}^{\mathrm{i}\lambda} p^n)(a_{1,n} - \mathrm{i}b_{1,n}) - F_2(\mathrm{e}^{\mathrm{i}\lambda} p^n)(a_{2,n} - \mathrm{i}b_{2,n}) \tag{2.3}$$

with the complex functions

$$F_1(z) := (1 + z^2)/\mathrm{i}(1 - z^2), \qquad F_2(z) := 2z/\mathrm{i}(1 - z^2). \tag{2.4}$$

For $\lambda = 0$ the formula (2.3) reduces to

$$\alpha_n = -\left((1 + p^{2n})b_{1,n} - 2p^n b_{2,n}\right)/(1 - p^{2n}),$$
$$\beta_n = \left((1 + p^{2n})a_{1,n} - 2p^n a_{2,n}\right)/(1 - p^{2n}).$$

Comparison with the formulae in [3], top of p. 200, shows that $K_{p,0}$ is the conjugation operator on the annulus R_p.

We denote by $\mathscr{B}(B_1, B_2)$ the set of bounded linear operators $A : B_1 \to B_2$ of a Banach space B_1 into another Banach space B_2. With the usual operator norm $\mathscr{B}(B_1, B_2)$ is itself a Banach space. The following lemma generalizes the first part of Theorem 3.1 of [3, p. 195]

Lemma 1. *For $0 < p < 1$ the operator K_p, λ is in $\mathscr{B}(L_r^2 \times L_r^2, L_r^2)$ as well as in $\mathscr{B}(W_r \times W_r, W_r)$. The inequality*

$$\|K_{p,\lambda}(\phi_1, \phi_2)\| \leqslant \left((1 + p^2)\|\phi_1\| + 2p\|\phi_2\|\right)/(1 - p^2) \tag{2.5}$$

holds in the L^2-norm as well as in the W-norm.

Proof. We observe that $|F_1(z)| \leqslant (1 + p^2)/(1 - p^2)$ and $|F_2(z)| \leqslant 2p/(1 - p^2)$ for $|z| \leqslant p$ and obtain from (2.3) the estimate

$$|\alpha_n - \mathrm{i}\beta_n| \leqslant \left((1 + p^2)|a_{1,n} - \mathrm{i}b_{1,n}| + 2p|a_{2,n} - \mathrm{i}b_{2,n}|\right)/(1 - p^2). \tag{2.6}$$

If ϕ_1, ϕ_2 are in L_r^2, then it follows from (2.6) that $\Sigma(\alpha_n^2 + \beta_n^2)$ is finite, hence $K_{p,\lambda}(\phi_1, \phi_2)$ is also in L_r^2. Summing the squares of (2.6) and applying the Schwarz inequality yields (2.5) for the L^2-norm. If ϕ_1, ϕ_2 are in W_r, then the sums $\Sigma n^2(a_{\nu,n}^2 + b_{\nu,n}^2)$ are finite. It follows from (2.6) that $\Sigma n^2(\alpha_n^2 + \beta_n^2)$ is also finite. Therefore the series

$$\sum_{n=1}^{\infty} n(\beta_n \cos nt - \alpha_n \sin nt) \tag{2.7}$$

represents an element in L_r^2. Comparison shows that (2.2) is an integral of the function (2.7). It follows that $K_{p,\lambda}(\phi_1, \phi_2)$ is in W_r and

$$\mathrm{d}K_{p,\lambda}(\phi_1, \phi_2)/\mathrm{d}t = K_{p,\lambda}(\phi_1', \phi_2').$$

The L^2-norm of the right-hand side can be estimated by the already proven L^2-norm version of (2.5). Since the Fourier series (2.2) has no constant term, its Sobolev norm is equal to the L^2-norm of its derivative. So it follows that (2.5) is also true for the W-norm. □

Now we investigate how $K_{p,\lambda}$ depends on the parameter p. To this end we introduce the operator

$$K'_{p,\lambda}(\phi_1, \phi_2)(t) \sim \sum_{n=1}^{\infty} \alpha'_n \cos nt + \beta'_n \sin nt. \tag{2.8}$$

The coefficients α'_n, β'_n are real numbers defined by

$$\alpha'_n - \mathrm{i}\beta'_n = n\, \mathrm{e}^{\mathrm{i}\lambda} p^{n-1}\big(F'_1(\mathrm{e}^{\mathrm{i}\lambda}p^n)(a_{1,n} - \mathrm{i}b_{1,n}) - F'_2(\mathrm{e}^{\mathrm{i}\lambda}p^n)(a_{2,n} - \mathrm{i}b_{2,n})\big) \tag{2.9}$$

in terms of the coefficients in the Fourier series (2.1) of the ϕ_ν and the derivatives F'_ν of the functions F_ν of (2.4), i.e.

$$F'_1(z) = 4z/\mathrm{i}(1 - z^2)^2, \qquad F'_2(z) = 2(1 + z^2)/\mathrm{i}(1 - z^2)^2. \tag{2.10}$$

Lemma 2. *The operator $K'_{p,\lambda}$ is in $\mathscr{B}(L^2_r \times L^2_r, L^2_r)$ as well as in $\mathscr{B}(W_r \times W_r, W_r)$, and the inequality*

$$\|K'_{p,\lambda}(\phi_1, \phi_2)\| \leqslant \frac{2}{\mathrm{e}\,|\log p|\, p(1 - p^2)^2}\big(2p\|\phi_1\| + (1 + p^2)\|\phi_2\|\big) \tag{2.11}$$

holds in the norm of L^2 as well as in the norm of W.

Proof. For $\mu > 0$ the real function $f(x) := x\mathrm{e}^{-\mu x}$ is non-negative for $x \geqslant 0$ with maximum at $x = 1/\mu$. Therefore $|f(x)| \leqslant 1/(\mathrm{e}\mu)$ for $x \geqslant 0$. We apply this for $\mu = -\log p$ and $x = n$ and obtain

$$np^n \leqslant 1/(\mathrm{e}\,|\log p|) \quad \text{for } n = 1, 2, 3, \ldots .$$

Furthermore $|F'_1(z)| \leqslant 4p/(1 - p^2)^2$ and $|F'_2(z)| \leqslant 2(1 + p^2)/(1 - p^2)^2$ for $|z| \leqslant p$. With these bounds for the factors in (2.9) the inequalities (2.11) can be proved like (2.5).

Lemma 3. *For fixed λ the mapping.*

$$(0, 1) \ni p \to K_{p,\lambda}$$

defines a differentiable curve in $\mathscr{B}(L^2_r \times L^2_r, L^2_r)$ as well as in $\mathscr{B}(W_r \times W_r, W_r)$. The derivative is

$$\mathrm{d}K_{p,\lambda}/\mathrm{d}p = K'_{p,\lambda} \tag{2.12}$$

with $K'_{p,\lambda}$ defined in (2.8).

Proof. For the purpose of this proof we express the dependence of the coefficients on p explicitly by writing $\alpha_n(p)$ and $\beta_n(p)$. Then it follows from (2.3) and (2.9) that

$$(\alpha_n(r) - \alpha_n(p))/(r - p) \to \alpha'_n(p) \quad \text{for } r \to p$$

and the same for β_n. For fixed p, the convergence is uniform with respect to n. Hence the same arguments which were used in the preceding lemmas also yield (2.12). □

It follows from Lemma 2 that in each interval $0 < \epsilon \leqslant p \leqslant 1 - \epsilon$ the norm of $K'_{p,\lambda}$ is uniformly bounded. So we obtain from Lemma 3 the

Corollary. *For each $\epsilon > 0$ there exists a constant C_ϵ such that*

$$\|K_{p,\lambda} - K_{r,\lambda}\| \leqslant C_\epsilon\, |p - r| \quad \text{for } \epsilon \leqslant p,\ r \leqslant 1 - \epsilon \tag{2.13}$$

holds in the norm of $\mathscr{B}(L^2_r \times L^2_r, L^2_r)$ as well as in the norm of $\mathscr{B}(W_r \times W_r, W_r)$.

In the special case $\lambda = 0$, the estimate (2.13) follows from the second part of Theorem 3.1 of [3, p. 195].

3. Boundary value problems

Lemma 4. *Let ϕ_1, ϕ_2 be in W_r and define*

$$\psi_1 := K_{p,\lambda}(\phi_1, \phi_2), \qquad \psi_2 := -K_{p,\lambda}(\phi_2, \phi_1).$$

Denote by $\hat{\phi}_\nu$ the mean values

$$\hat{\phi}_\nu := \frac{1}{2\pi}\int_0^{2\pi} \phi_\nu(t)\mathrm{d}t.$$

Then there exists a function Ψ analytic in the annulus $R_p = \{z : p < |z| < 1\}$ and continuous in the closure $\overline{R}_p$ with boundary values

$$\Psi(\mathrm{e}^{\mathrm{i}t}) = \phi_1(t) + \mathrm{i}\psi_1(t) - \hat{\phi}_1, \tag{3.1}$$

$$\Psi(p\,\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{-\mathrm{i}\lambda}(\phi_2(t) + \mathrm{i}\psi_2(t) - \hat{\phi}_2). \tag{3.2}$$

Proof. Define the coefficients $d_0 := 0$ and

$$\begin{aligned} d_n &:= (A_{1,n} - \mathrm{e}^{\mathrm{i}\lambda}p^n A_{2,n})/(1 - \mathrm{e}^{2\mathrm{i}\lambda}p^{2n}) \\ d_{-n} &:= (\overline{A}_{1,n} - \mathrm{e}^{\mathrm{i}\lambda}p^{-n}\overline{A}_{2,n})/(1 - \mathrm{e}^{2\mathrm{i}\lambda}p^{-2n}) \end{aligned} \tag{3.3}$$

for $n \geqslant 1$ with $A_{\nu,n} := a_{\nu,n} - \mathrm{i}b_{\nu,n}$, and the formal Laurent series

$$\Psi(z) := \sum_{n=-\infty}^{+\infty} d_n z^n. \tag{3.4}$$

It follows from (3.3) that

$$\max(|d_n|, p^{-n}|d_{-n}|) \leqslant (|A_{1,n}| + |A_{2,n}|)/(1-p^2)$$

holds for $n = 1, 2, \ldots$. Since the functions ϕ_ν are in W, the series $\Sigma|A_{\nu,n}|$ converge. The estimates for $d_{\pm n}$ then show that the series (3.4) converges uniformly for $p \leqslant |z| \leqslant 1$.

It follows that (3.4) represents a function Ψ which is analytic in R_p and continuous in the closure. Comparison of the series (3.4) evaluated for $z = \mathrm{e}^{\mathrm{i}t}$ and $z = p\mathrm{e}^{\mathrm{i}t}$ with (2.1) and (2.2) shows that Ψ has the boundary values (3.1) and (3.2). □

The function Ψ defined in (3.4) depends on p. One can form the function

$$\frac{\partial \Psi}{\partial p}(z) = \sum_{n=-\infty}^{+\infty} \frac{\partial d_n}{\partial p} z^n \tag{3.5}$$

which is analytic in R_p. The derivative Ψ' is represented by the Laurent series

$$\Psi'(z) = \sum_{n=-\infty}^{+\infty} n d_n z^{n-1}. \tag{3.6}$$

Since ϕ_1, ϕ_2 are in W, the series (3.5) and (3.6) also converge in L^2 at the boundary. Hence one can define

$$\Psi'(r\,\mathrm{e}^{\mathrm{i}t}) = \sum_{n=-\infty}^{+\infty} n d_n r^{n-1} \mathrm{e}^{\mathrm{i}(n-1)t}$$

for $r = p$ and $r = 1$ as L^2-functions, and analogously with the boundary values of $\partial\Psi/\partial p$. By comparison of the series, it follows that

$$\frac{\partial \Psi}{\partial p}(\mathrm{e}^{\mathrm{i}t}) = \mathrm{i} K'_{p,\lambda}(\phi_1, \phi_2), \tag{3.7}$$

$$\frac{\partial \Psi}{\partial p}(p\,\mathrm{e}^{\mathrm{i}t}) + \mathrm{e}^{\mathrm{i}t}\Psi'(\mathrm{e}^{\mathrm{i}t}) = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda} K'_{p,\lambda}(\phi_2, \phi_1) \tag{3.8}$$

hold in L^2, hence for almost all t. We know from Lemma 2 that the right-hand sides of (3.7) and (3.8) are in W. Hence one can also choose the left-hand sides as continuous functions. Then (3.7) and (3.8) hold for all t.

Now we are able to discuss the following problem of generalized conjugation.

Problem K. For given real functions $\phi_1, \phi_2 \in W$ and real numbers p, λ with $0 < p < 1$, determine a function Ψ analytic in R_p and continuous in $\overline{R}_p$ such that

$$\operatorname{Re} \Psi(\mathrm{e}^{\mathrm{i}t}) = \phi_1(t), \qquad \operatorname{Re}\big(\mathrm{e}^{\mathrm{i}\lambda}\Psi(p\,\mathrm{e}^{\mathrm{i}t})\big) = \phi_2(t).$$

Lemma 5. (a) *If* $\sin\lambda \neq 0$, *then problem* K *has a unique solution* Ψ *for any* $\phi_1, \phi_2 \in W_r$. *The boundary values of* Ψ *are given by*

$$\Psi(\mathrm{e}^{\mathrm{i}t}) = \phi_1 + \mathrm{i} K_{p,\lambda}(\phi_1, \phi_2) + \mathrm{i}\alpha, \tag{3.9}$$

$$\mathrm{e}^{\mathrm{i}\lambda}\Psi(p\,\mathrm{e}^{\mathrm{i}t}) = \phi_2 - \mathrm{i} K_{p,\lambda}(\phi_2, \phi_1) - \hat{\phi}_2 + \mathrm{e}^{\mathrm{i}\lambda}(\hat{\phi}_1 + \mathrm{i}\alpha) \tag{3.10}$$

with $\alpha = (\hat{\phi}_1 \cos\lambda - \hat{\phi}_2)/\sin\lambda$.

(b) *If* $\sin\lambda = 0$, *then Problem K has a solution iff* $\hat{\phi}_2 = \hat{\phi}_1\cos\lambda$. *In this case the gen is also represented by* (3.9) *and* (3.10) *with* α *as a free parameter.*

Proof. It follows from Lemma 4 that there exists a function Ψ_0 analytic in R_p and continuous in $\overline{R}_p$ which satisfies

$$\operatorname{Re} \Psi_0(\mathrm{e}^{\mathrm{i}t}) = \phi_1 - \hat{\phi}_1, \qquad \operatorname{Re}\big(\mathrm{e}^{\mathrm{i}\lambda}\Psi_0(p\,\mathrm{e}^{\mathrm{i}t})\big) = \phi_2 - \hat{\phi}_2.$$

Let Ψ be a solution of Problem K. Then $\Psi_1 := \Psi - \Psi_0$ is a function analytic in R_p and continuous in $\overline{R}_p$ which satisfies

$$\operatorname{Re} \Psi_1(\mathrm{e}^{\mathrm{i}t}) = \hat{\phi}_1, \qquad \operatorname{Re}\big(\mathrm{e}^{\mathrm{i}\lambda}\Psi_1(p\,\mathrm{e}^{\mathrm{i}t})\big) = \hat{\phi}_2.$$

This means that Ψ_1 maps R_p onto a region whose boundary is contained in the union of the two straight lines $H_1 := \{z \in \mathbb{C}\colon \operatorname{Re} z = \hat{\phi}_1\}$ and $H_2 := \{z \in \mathbb{C}\colon \operatorname{Re}(\mathrm{e}^{\mathrm{i}\lambda} z) = \hat{\phi}_2\}$. Since Ψ_1 is a bounded analytic function this is possible only if $\Psi_1 = d$ is constant with $d \in H_1 \cap H_2$. In case (a) the two lines are not parallel and the point $d = \hat{\phi}_1 + \mathrm{i}(\hat{\phi}_1\cos\lambda - \hat{\phi}_2)/\sin\lambda$ of intersection is uniquely determined. In case (b) the lines are parallel and a common point d exists only if $H_1 = H_2$. Then $d = \hat{\phi}_1 + \mathrm{i}\alpha$ with $\alpha \in \mathbb{R}$ as a free parameter. □

In the special case $\lambda = 0$, Problem K reduces to ordinary conjugation on the annulus. This problem is discussed in [3, p. 194 ff] in connection with the method of Theodorsen–Garrick, and case (b) of Lemma 5 corresponds to Theorem 3.2 of [3, p. 198].

We are now in position to treat the Riemann–Hilbert problem on an annulus.

Problem RH. Given a number p, $0 < p < 1$, complex-valued functions $A_\nu \in W$, $A_\nu \neq 0$, and real functions $f_\nu \in W$, determine a function Ψ analytic in R_p and continuous in $\bar{R}_p$ such that

$$\operatorname{Re}(A_1(t)\Psi(e^{it})) = f_1(t), \qquad \operatorname{Re}(A_2(t)\Psi(p\,e^{it})) = f_2(t). \tag{3.11}$$

The Riemann–Hilbert problem for multiply connected regions is discussed by Vekua in [5] and by Bojarski (also in [5]). In [1] Banzuri gives an explicit solution of problem (RH) in terms of certain singular convolution integrals. Here we pursue a different approach and give a solution of Problem RH in closed form in terms of the operators $K_{p,\lambda}$.

Since the functions A_ν do not vanish, they can be written in the form

$$A_\nu(t) = \rho_\nu(t)\exp(i\beta_\nu(t)) \tag{3.12}$$

with $\rho_\nu > 0$ and real continuous functions β_ν. There exist integers m_1 and m_2 such that

$$v_\nu(t) := \beta_\nu(t) - m_\nu \cdot t \tag{3.13}$$

are 2π-periodic continuous functions. We restrict our attention to the special case in which the winding numbers of A_1 and A_2 are equal, i.e. we assume

$$m_1 = m_2 = m. \tag{3.14}$$

There exists $c > 0$ such that $|A_\nu(t)| \geq c$. Therefore the functions β_ν are also absolutely continuous with square-integrable derivatives. Hence the functions v_ν are in W_r and the functions

$$w_1 = K_{p,0}(v_1, v_2), \qquad w_2 = -K_{p,0}(v_2, v_1) \tag{3.15}$$

are also in W_r. It follows from Lemma 4 that there exists a function V analytic in R_p with boundary values

$$V(e^{it}) = w_1(t) - iv_1(t),$$
$$V(p\,e^{it}) = w_2(t) - iv_2(t) + i\lambda$$

with the number λ defined by the difference

$$\lambda = \hat{v}_2 - \hat{v}_1 \tag{3.16}$$

of the mean values $\hat{v}_\nu$ of the v_ν. If we insert the Ansatz

$$\Psi(z) = z^{-m}H(z)\exp(V(z)) \tag{3.17}$$

into (3.11), the factors $\exp(\pm i\beta_\nu(t))$ and $\exp(\pm imt)$ cancel, and we obtain for the analytic function H a conjugation Problem K

$$\operatorname{Re} H(e^{it}) = g_1(t) := \frac{f_1(t)}{\rho_1(t)}\exp(-w_1(t)),$$

$$\operatorname{Re}(e^{i\lambda}H(p\,e^{it})) = g_2(t) := p^m\frac{f_2(t)}{\rho_2(t)}\exp(-w_2(t)).$$

We note that the functions g_ν are in W. Therefore, we obtain from Lemma 5 the following.

Lemma 6. Assume that the winding numbers of A_1 and A_2 are equal. Then the following holds true:

(a) If $\sin \lambda \neq 0$, then Problem RH has a unique solution. The boundary values of the solution are

$$\Psi(\mathrm{e}^{\mathrm{i}t}) = \left(g_1 + \mathrm{i}K_{p,\lambda}(g_1, g_2) + \mathrm{i}\alpha\right)\exp(w_1 - \mathrm{i}\beta_1), \tag{3.18}$$

$$\Psi(p\,\mathrm{e}^{\mathrm{i}t}) = \left(g_2 - \mathrm{i}K_{p,\lambda}(g_2, g_1) + \mathrm{e}^{\mathrm{i}\lambda}(\hat{g}_1 + \mathrm{i}\alpha) - \hat{g}_2\right)p^{-m}\exp(w_2 - \mathrm{i}\beta_2) \tag{3.19}$$

with the real constant

$$\alpha = (\hat{g}_1 \cos \lambda - \hat{g}_2)/\sin \lambda. \tag{3.20}$$

(b) If $\sin \lambda = 0$, the problem has a solution iff $\hat{g}_2 = \hat{g}_1 \cos \lambda$. In this case the boundary values of the solution are also represented by (3.18) and (3.19) with $\alpha \in \mathbb{R}$ as a free parameter.

4. The iterative method

For this section we assume that the functions η_ν have Lipschitz-continuous first derivatives $\dot{\eta}_\nu \neq 0$. Suppose that we have approximations $S_{\nu,k}$ for the functions S_ν, $\nu = 1, 2$, and q_k for the number q, such that $S_{\nu,k}(t) - t \in W_r$ and $0 < q_k < 1$. We want to find new functions

$$S_{\nu,k+1}(t) = S_{\nu,k}(t) + U_{\nu,k}(t) \tag{4.1}$$

and a new number q_{k+1} which are in some sense better approximations. In view of (1.1), it would be best to construct the $S_{\nu,k+1}$ and q_{k+1} in such a way that there exists a function Φ_{k+1} which is analytic in $q_{k+1} < |z| < 1$, continuous for $q_{k+1} \leqslant |z| \leqslant 1$, and has the boundary values

$$\Phi_{k+1}(\mathrm{e}^{\mathrm{i}t}) = \eta_1(S_{1,k+1}(t)), \qquad \Phi_{k+1}(q_{k+1}\mathrm{e}^{\mathrm{i}t}) = \eta_2(S_{2,k+1}(t)). \tag{4.2}$$

Then Φ_{k+1} would already be the correct conformal mapping Φ. In practice we must be content to satisfy (4.2) as closely as possible. A reasonable goal is to fulfill (4.2) at least through the terms of first order in the shifts $U_{\nu,k}$.

This approximation to (4.2) can be formulated in the following way: Search for a number q_{k+1} and real function $U_{\nu,k}$ such that there exists for $p = q_{k+1}$ a function Φ_{k+1} analytic for $p \leqslant |z| \leqslant 1$ with boundary values

$$\Phi_{k+1}(\mathrm{e}^{\mathrm{i}t}) = \eta_1(S_{1,k}(t)) + \dot{\eta}_1(S_{1,k}(t))U_{1,k}(t), \tag{4.3}$$

$$\Phi_{k+1}(p\,\mathrm{e}^{\mathrm{i}t}) = \eta_2(S_{2,k}(t)) + \dot{\eta}_2(S_{2,k}(t))U_{2,k}(t). \tag{4.4}$$

The tangent angle θ_ν of the curve Γ_ν is defined by

$$\dot{\eta}_\nu(s) = \rho_\nu(s)\exp(\mathrm{i}\theta_\nu(s)).$$

The condition that the $U_{\nu,k}$ are real functions is equivalent to the boundary conditions

$$\begin{aligned} &\operatorname{Im}\left(\exp(-\mathrm{i}\theta_1(S_{1,k}(t)))\Phi_{k+1}(\mathrm{e}^{\mathrm{i}t})\right) = f_1(t), \\ &\operatorname{Im}\left(\exp(-\mathrm{i}\theta_2(S_{2,k}(t)))\Phi_{k+1}(p\,\mathrm{e}^{\mathrm{i}t})\right) = f_2(t) \end{aligned} \tag{4.5}$$

for the analytic function Φ_{k+1}. The right-hand sides are defined by

$$f_\nu(t) := \operatorname{Im}\left(\exp(-\mathrm{i}\theta_\nu(S_{\nu,k}(t)))\eta_\nu(S_{\nu,k}(t))\right). \tag{4.6}$$

In order to take care of the normalization (1.2) we assume that $S_{1,k}(0) = 0$ and demand that

$$\Phi_{k+1}(1) = \eta_1(0),$$

which is then equivalent to

$$U_{1,k}(0) = 0. \tag{4.7}$$

Since $\dot\eta_\nu$ is Lipschitz-continuous, $\dot\eta_\nu(S_{\nu,k}(t))$ is in W. Therefore the coefficients $\exp(-\mathrm{i}\theta_\nu(S_{\nu,k}(t)))$ and the right-hand sides f_ν in (4.5) have the regularity which is required in Problem RH.

By the substitution $\Phi_{k+1} = \mathrm{i}\Psi$, the conditions (4.5) are transformed into a Problem RH for the function Ψ. The method described in the last section to calculate a solution Ψ of this Problem RH can therefore be used to obtain a solution of (4.5). We observe that $\beta_\nu(t) = -\theta_\nu(S_{\nu,k}(t))$. Therefore $m = -1$. According to (3.13) ff. we have to calculate

$$v_\nu(t) := t - \theta_\nu(S_{\nu,k}(t)), \tag{4.8}$$

$$w_1 := K_{p,0}(v_1, v_2), \qquad w_2 := -K_{p,0}(v_2, v_1), \tag{4.9}$$

$$\lambda := \hat v_2 - \hat v_1, \tag{4.10}$$

$$g_1(t) := f_1(t)\exp(-w_1(t)), \qquad g_2(t) := \frac{1}{p} f_2(t)\exp(-w_2(t)), \tag{4.11}$$

$$h_1 = K_{p,\lambda}(g_1, g_2), \qquad h_2 = -K_{p,\lambda}(g_2, g_1) \tag{4.12}$$

and we finally arrive at

$$\Phi_{k+1}(\mathrm{e}^{\mathrm{i}t}) = (\mathrm{i}g_1(t) - h_1(t) - \alpha)\exp(w_1(t) + \mathrm{i}\theta_1(S_{1,k}(t))), \tag{4.13}$$

$$\Phi_{k+1}(p\,\mathrm{e}^{\mathrm{i}t}) = (\mathrm{i}g_2(t) - h_2(t) + \mathrm{e}^{\mathrm{i}\lambda}(\mathrm{i}\hat g_1 - \alpha) - \mathrm{i}\hat g_2)p\exp(w_2(t) + \mathrm{i}\theta_2(S_{2,k}(t))). \tag{4.14}$$

We see from (4.3), (4.4), (4.13), (4.14) that the shifts $U_{\nu,k}$ must be calculated by the following formulas

$$U_{1,k}(t) = -(h_1(t) + \alpha)\frac{\exp(w_1(t))}{|\dot\eta_1(S_{1,k}(t))|} - \mathrm{Re}\left(\frac{\eta_1(S_{1,k}(k))}{\dot\eta_1(S_{1,k}(t))}\right), \tag{4.15}$$

$$U_{2,k}(t) = -(h_2(t) + \alpha\cos\lambda + \hat g_1 \sin\lambda)\frac{p\exp(w_2(t))}{|\dot\eta_2(S_{2,k}(t))|} - \mathrm{Re}\left(\frac{\eta_2(S_{2,k}(t))}{\dot\eta_2(S_{2,k}(t))}\right). \tag{4.16}$$

The normalization condition (4.7) then specifies the parameter α:

$$\alpha = -\mathrm{Re}(\exp(-w_1(0) - \mathrm{i}\theta_1(0))\eta_1(0)) - h_1(0). \tag{4.17}$$

In general this choice of α is in conflict with (3.20). Therefore we must not expect that with α defined by (4.17) the function Φ_{k+1} with boundary values (4.13) and (4.14) is a solution of (4.5). In fact one calculates easily that Φ_{k+1} satisfies the outer boundary condition in (4.5) but not the inner one. Instead one obtains

$$\mathrm{Im}(\exp(-\mathrm{i}\theta_2(S_{2,k}(t)))\Phi_{k+1}(p\,\mathrm{e}^{\mathrm{i}t})) = f_2(t) + p\gamma(p)\exp(w_2(t)) \tag{4.18}$$

with the real number

$$\gamma(p) := \hat g_1\cos\lambda - \hat g_2 - \alpha\sin\lambda. \tag{4.19}$$

Hence the functions Φ_{k+1} and $U_{\nu,k}$ as defined in (4.13)–(4.16) satisfy (4.3), but instead of (4.4),

$$\begin{aligned}\Phi_{k+1}(p\,\mathrm{e}^{\mathrm{i}t}) = \eta_2(S_{2,k}(t)) + \dot\eta_2(S_{2,k}(t))U_{2,k}(t)\\ +p\gamma(p)\exp(w_2(t))\mathrm{i}\exp(\mathrm{i}\theta_2(S_{2,k}(t))).\end{aligned} \tag{4.20}$$

Therefore at the inner boundary $\Phi_{k+1}(p\mathrm{e}^{\mathrm{i}t})$ is obtained from $\eta_2(S_{2,k}(t))$ not only by a tangential shift, but also by a shift in the direction of the inner normal i $\exp(\mathrm{i}\theta_2(S_{2,k}(t)))$. Along the whole curve Γ_2 this normal shift has constant sign, which is determined only by the sign of γ. It tries to make the inner hole in the region G smaller in the case of positive γ and larger in the case of negative γ. This must be interpreted as a reaction to an inappropriate choice of p. The quantity $\gamma(p)$ is a measure of the error if the inverse module $q = 1/M$ of G is approximated by p. Therefore it is a reasonable strategy to choose the new approximation q_{k+1} for q in such a way that $\gamma(q_{k+1}) = 0$. We retain the order of our approximation if we perform only one step of a Newton method for this equation. This leads to the prescription

$$q_{k+1} = q_k - \gamma(q_k)/\gamma'(q_k) \tag{4.21}$$

with $\gamma'(p) = \mathrm{d}\gamma/\mathrm{d}p$. One can express this derivative easily in terms of the operator K' defined in (2.8). We observe that

$$\frac{\partial w_1}{\partial p} = K'_{p,0}(v_1, v_2), \qquad \frac{\partial w_2}{\partial p} = -K'_{p,0}(v_2, v_1), \tag{4.22}$$

$$\frac{\partial g_1}{\partial p} = -g_1\frac{\partial w_1}{\partial p}, \qquad \frac{\partial g_2}{\partial p} = -g_2\left(\frac{\partial w_2}{\partial p} + \frac{1}{p}\right), \tag{4.23}$$

$$\frac{\partial h_1}{\partial p} = K'_{p,\lambda}(g_1, g_2) + K_{p,\lambda}\left(\frac{\partial g_1}{\partial p}, \frac{\partial g_2}{\partial p}\right), \tag{4.24}$$

and obtain

$$\frac{\partial\alpha}{\partial p} = \frac{\partial w_1(0)}{\partial p}\mathrm{Re}(\exp(-w_1(0) - \mathrm{i}\theta_1(0))\eta_1(0)) - \frac{\partial h_1(0)}{\partial p}, \tag{4.25}$$

$$\frac{\partial\gamma}{\partial p} = \frac{1}{2\pi}\int\left(\frac{\partial g_1}{\partial p}\cos\lambda - \frac{\partial g_2}{\partial p}\right)\mathrm{d}t - \frac{\partial\alpha}{\partial p}\sin\lambda. \tag{4.26}$$

Now we calculate some of these quantities for the correct conformal mapping, i.e. we put $q_k = q$, $S_{\nu,k} = S_\nu$. Differentiating the identities (1.1) yields

$$\begin{aligned}\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\Phi'(\mathrm{e}^{\mathrm{i}t}) = \dot\eta_1(S_1(t))S_1'(t),\\ q\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\Phi'(q\,\mathrm{e}^{\mathrm{i}t}) = \dot\eta_2(S_2(t))S_2'(t).\end{aligned} \tag{4.27}$$

Φ is a conformal mapping. Hence the derivative Φ' does not vanish in R_q. Therefore we can take the logarithm of (4.27) and obtain (up to multiplies of $2\pi\mathrm{i}$)

$$\begin{aligned}\log\Phi'(\mathrm{e}^{\mathrm{i}t}) = \log|\dot\eta_1(S_1(t))S_1'(t)| + \mathrm{i}(\theta_1(S_1(t)) - t - \tfrac{1}{2}\pi),\\ \log\Phi'(q\,\mathrm{e}^{\mathrm{i}t}) = \log|\dot\eta_2(S_2(t))S_2'(t)/q| + \mathrm{i}(\theta_2(S_2(t)) - t - \tfrac{1}{2}\pi).\end{aligned} \tag{4.28}$$

The right-hand sides are 2π-periodic functions. Hence $\log\Phi'(z)$ is a single-valued analytic function in R_q. If v_ν and w_ν are defined as in (4.8) and (4.9) with $p = q$ and $S_{\nu,k} = S_\nu$, then it follows from (4.28) and Cauchy's theorem that $\lambda \equiv 0 \pmod{2\pi}$. Comparison of Lemma 4 with

(4.28) shows that

$$\begin{aligned} \mathrm{i}\Phi'(\mathrm{e}^{\mathrm{i}t}) &= D_0 \exp(w_1(t) - \mathrm{i}v_1(t)), \\ \mathrm{i}\Phi'(q\,\mathrm{e}^{\mathrm{i}t}) &= D_0 \exp(w_2(t) - \mathrm{i}v_2(t)) \end{aligned} \tag{4.29}$$

with a positive real constant D_0. In order to calculate this number, we take the absolute value in (4.29), which yields

$$|\Phi'(\mathrm{e}^{\mathrm{i}t})| = D_0 \exp(w_1(t)). \tag{4.30}$$

Now we integrate over $[0, 2\pi]$. We observe that $t \to \Phi(\mathrm{e}^{\mathrm{i}t})$ is a parametrization of the curve Γ_1. Therefore the integral of the left-hand side in (4.30) is equal to the total arc length $L(\Gamma_1)$ of the curve Γ_1. Hence we obtain

$$D_0 = L(\Gamma_1) \Big/ \int_0^{2\pi} \exp(w_1(t))\,\mathrm{d}t. \tag{4.31}$$

Now we calculate the functions g_ν. We insert the definition (4.6) of f_ν with $S_{\nu,k} = S_\nu$ into (4.11), use (1.1), and replace the exponential term in view of (4.29) by the derivative Φ'. Thus we obtain

$$g_1(t) = \operatorname{Re} F(\mathrm{e}^{\mathrm{i}t}), \qquad g_2(t) = \operatorname{Re} F(q\,\mathrm{e}^{\mathrm{i}t}) \tag{4.32}$$

with the analytic function

$$F(z) = -D_0 \Phi(z)/z\Phi'(z). \tag{4.33}$$

Since $\lambda = 0$, we obtain for $p = q$

$$\begin{aligned} \frac{\partial \gamma}{\partial q} &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial g_1}{\partial q} - \frac{\partial g_2}{\partial q} \right) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(g_2 \left(\frac{\partial w_2}{\partial q} + \frac{1}{q} \right) - g_1 \frac{\partial w_1}{\partial q} \right) \mathrm{d}t. \end{aligned} \tag{4.34}$$

The derivatives $\partial w_\nu / \partial q$ are expressed by (4.22) in terms of the operator $K'_{p,0}$. These formulas can be evaluated by means of (3.7) and (3.8). In view of (4.28), the function $\Psi(z)$ in (3.8) is equal to $\mathrm{i} \log \Phi'(z) + C$. We conclude that there exists an analytic function $F_0(z)$ in R_q such that

$$\frac{\partial w_1}{\partial q} = F_0(\mathrm{e}^{\mathrm{i}t}), \qquad \frac{\partial w_2}{\partial q} = F_0(q\,\mathrm{e}^{\mathrm{i}t}) + \mathrm{e}^{\mathrm{i}t} \frac{\Phi''(q\,\mathrm{e}^{\mathrm{i}t})}{\Phi'(q\,\mathrm{e}^{\mathrm{i}t})}. \tag{4.35}$$

We replace the functions g_ν and $\partial w_\nu | \partial q$ in (4.34) by (4.32) and (4.35):

$$\begin{aligned} \frac{\partial \gamma}{\partial q} &= \frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} \left(F(q\,\mathrm{e}^{\mathrm{i}t}) \left(\frac{\partial w_2}{\partial q} + \frac{1}{q} \right) - F(\mathrm{e}^{\mathrm{i}t}) \frac{\partial w_1}{\partial q} \right) \mathrm{d}t \\ &= \frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} \left(F(q\,\mathrm{e}^{\mathrm{i}t}) F_0(q\,\mathrm{e}^{\mathrm{i}t}) - F(\mathrm{e}^{\mathrm{i}t}) F_0(\mathrm{e}^{\mathrm{i}t}) \right) \mathrm{d}t \\ &\quad + \frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} F(q\,\mathrm{e}^{\mathrm{i}t}) \left(\frac{1}{q} + \mathrm{e}^{\mathrm{i}t} \frac{\Phi''(q\,\mathrm{e}^{\mathrm{i}t})}{\Phi'(q\,\mathrm{e}^{\mathrm{i}t})} \right) \mathrm{d}t. \end{aligned} \tag{4.36}$$

The first integral vanishes in view of Cauchy's theorem. For the second part of the last integral we use the definition (4.33) of F and integrate by parts:

$$-D_0\int\frac{\Phi(q\,\mathrm{e}^{\mathrm{i}t})\Phi''(q\,\mathrm{e}^{\mathrm{i}t})}{q(\Phi'(q\,\mathrm{e}^{\mathrm{i}t}))^2}\mathrm{d}t=-\frac{2\pi D_0}{q}+\frac{D_0}{q^2}\int\frac{\Phi(q\,\mathrm{e}^{\mathrm{i}t})}{\mathrm{e}^{\mathrm{i}t}\Phi'(q\,\mathrm{e}^{\mathrm{i}t})}\mathrm{d}t$$

$$=-\frac{D_0}{q}2\pi-\frac{1}{q}\int F(q\,\mathrm{e}^{\mathrm{i}t})\mathrm{d}t.$$

We see that the last summand cancels the remaining term in (4.36), and obtain finally the simple expression

$$\partial\gamma/\partial q=-D_0/q. \tag{4.37}$$

Now by insertion of the formula (4.31) for D_0, we see finally that $\gamma'(q)=\tilde\gamma'(q)$, where $\tilde\gamma'$ is defined by the expression

$$\tilde\gamma'(q)=-L(\Gamma_1)/q\int_0^{2\pi}\exp(w_1(t))\mathrm{d}t. \tag{4.38}$$

The right-hand side of (4.38) is nonzero for all values of q in $0<q<1$ and functions w_1. Therefore $\gamma'(q)\neq 0$ for the correct solution. We can use (4.38) as an approximation for the γ' in (4.21), which yields

$$q_{k+1}=q_k\left(1+\gamma(q_k)\frac{1}{L(\Gamma_1)}\int_0^{2\pi}\exp(w_1(t))\mathrm{d}t\right). \tag{4.39}$$

The iterative method is now as follows:

Step 1. Insert the given functions $S_{\nu,k}$ into θ_ν and calculate the functions v_ν as in (4.8). Put $p=q_k$ and calculate the functions w_ν from (4.9) and the number λ from (4.10).
Step 2. By inserting $S_{\nu,k}$ into θ_ν and η_ν, determine f_ν and g_ν according to (4.6) and (4.11). Then calculate the functions h_ν from (4.12) and the number α from (4.17).
Step 3. Determine the number $\gamma(q_k)$ from (4.19) and calculate the new q_{k+1} from (4.39).
Step 4. Put $p=q_{k+1}$ and calculate the functions w_ν, g_ν, h_ν again with this new value of p.
Step 5. Determine the functions $U_{\nu,k}$ from (4.15), (4.16) and the new functions $S_{\nu,k+1}$ from (4.1).

The operators $K_{p,\lambda}$ are defined for functions in W_r and yield functions in W_r. It follows that $U_{\nu,k}$ and $S_{\nu,k+1}(t)-t$ are in W_r. Therefore, if one starts the iteration with functions $S_{\nu,0}$ such that $S_{\nu,0}(t)-t$ is in W_r, it can be continued as long as q_k remains in the interval $0<q_k<1$.

5. Convergence

Theorem 1. *If the functions η_ν have second derivatves $\ddot\eta_\nu$ which satisfy a Hölder condition with exponent μ, then for each $\delta>0$ and $M>0$ there exists a number ϵ_0, which depends only on the functions η_ν and the numbers δ and M, with the following property:*

If q_0 is a number and $S_{\nu,0}$ are functions such that

$$\delta \leqslant q_0 \leqslant 1-\delta, \tag{5.1}$$

$$S_{1,0}(0)=0, \tag{5.2}$$

$$S_{\nu,0}(t)-t \in W_r, \qquad \|S'_{\nu,0}\|_2 \leqslant M, \tag{5.3}$$

$$|\gamma'(q_0)-\tilde{\gamma}'(q_0)| \leqslant \epsilon_0, \tag{5.4}$$

and there exists a function Φ_0 analytic for $q_0 < |z| < 1$ and continuous for $q_0 \leqslant |z| \leqslant 1$ such that $\Phi r(e^{it}) \in W$ for $r=q_0$ and $r=1$ and

$$\|\eta_1(S_{1,0}(t))-\Phi_0(e^{it})\|_W \leqslant \epsilon_0, \qquad \|\eta_2(S_{2,0}(t))-\Phi_0(q_0e^{it})\|_W \leqslant \epsilon_0, \tag{5.5}$$

then the iteration converges in W, i.e.

$$\|S_{\nu,k}-S_\nu\|_W \to 0$$

and $q_k \to q$. The order of convergence is at least $1+\mu$.

Proof. We have shown at the end of Section 4 that the iteration can be continued as long as $0<q_k<1$. The functions $S_{\nu,k}(t)-t$ are all in W_r. First we assume in addition that there exists a positive number $\delta_1 < \frac{1}{2}\delta$ such that

$$\|S_{\nu,k}-S_{\nu,0}\|_W \leqslant \delta_1, \tag{5.6}$$

$$|q_k-q_0| \leqslant \delta_1 \tag{5.7}$$

for all k. Then all functions which appear in the iterative process are bounded in W by some constant which depends on M. Assume that the number q_k, the functions $S_{\nu,k}$ and the function Φ_k are determined so that Φ_k is analytic for $q_k<|z|<1$ and continuous for $q_k \leqslant |z| \leqslant 1$, and the boundary values are functions in W which satisfy

$$\|\eta_1(S_{1,k}(t))-\Phi_k(e^{it})\|_W \leqslant \epsilon_k, \qquad \|\eta_2(S_{2,k}(t))-\Phi_k(q_ke^{it})\|_W \leqslant \epsilon_k. \tag{5.8}$$

Then for $p=q_k$ one can subtract Φ_k on both sides of (4.3) and (4.4), and one arrives at a Riemann–Hilbert problem with inhomogeneous terms

$$\begin{aligned} f_1^*(t) &= \mathrm{Im}\big(\exp(-\mathrm{i}\theta_1(S_{1,k}(t)))\big(\eta_1(S_{1,k}(t))-\Phi_k(e^{it})\big)\big),\\ f_2^*(t) &= \mathrm{Im}\big(\exp(-\mathrm{i}\theta_2(S_{2,k}(t)))\big(\eta_2(S_{2,k}(t))-\Phi_k(q_ke^{it})\big)\big). \end{aligned} \tag{5.9}$$

Now we solve the Riemann–Hilbert problem (4.5) for $p=q_k$ with f_ν^* as right-hand sides along the lines of formulas (4.8) ff., and distinguish the functions which appear here by a star. It follows from (4.11) ff. and Lemma 1 that

$$\begin{aligned} &\|g_\nu^*\|_W \leqslant C\epsilon_k, \quad \|h_\nu^*\|_W \leqslant C\epsilon_k, \qquad |\alpha^*| \leqslant C\epsilon_k,\\ &|\gamma^*(q_k)| \leqslant C\epsilon_k, \quad \|U_{\nu,k}^*\|_W \leqslant C\epsilon_k. \end{aligned} \tag{5.10}$$

It is easily calculated that the Φ_{k+1} derived from this modified problem satisfies (4.3) and (4.20) with $p=q_k$ and $U_{\nu,k}$ and γ_k replaced by the starred quantities. Hence $\gamma_k(q_k)=\gamma_k^*(q_k)$, which implies in view of (4.39) that

$$|q_{k+1}-q_k| \leqslant C|\gamma(q_k)| \leqslant C_1\epsilon_k. \tag{5.11}$$

Furthermore we conclude that one obtains the same $U_{\nu,k}^*$ from the Riemann–Hilbert problem with f_ν as right-hand sides. We continue to mark these quantities by stars in order to distinguish them from the final functions $U_{\nu,k}$ which are derived from a solution of (4.5) for $p = q_{k+1}$. Hence $U^*_{\nu,k}$ and $U_{\nu,k}$ are represented by the same set of formulas with $p = q_k$ and $p = q_{k+1}$, respectively. The number p appears only as a subscript in the operator $K_{p,\lambda}$ and as a factor in the representation for g_2 and $U_{2,k}$. Since $K_{p,\lambda}$ depends in a Lipschitz-continuous way on p by virtue of (2.13), one can conclude

$$\|U_{\nu,k} - U^*_{\nu,k}\|_W \leqslant C\,|q_{k+1} - q_k|.$$

In view of (5.10) and (5.11), this implies finally

$$\|U_{\nu,k}\|_W \leqslant C\epsilon_k. \tag{5.12}$$

These $U_{\nu,k}$ are derived from an approximate solution Φ_{k+1} of the Riemann–Hilbert problem (4.5). The relations (4.3) and (4.20) are satisfied for $p = q_{k+1}$. It follows from (4.3) that

$$\eta_1(S_{1,k+1}(t)) - \Phi_{k+1}(\mathrm{e}^{\mathrm{i}t}) = U_{1,k}(t)\cdot\int_0^1 (\dot\eta_1(S_{1,k}(t) + \tau U_{1,k}(t)) - \dot\eta_1(S_{1,k}(t)))\mathrm{d}\tau.$$

This equation can be differentiated with respect to t. In view of the Hölder continuity of $\ddot\eta_1$ it follows that

$$\|\eta_1(S_{1,k+1}(t)) - \Phi_{k+1}(\mathrm{e}^{\mathrm{i}t})\|_W \leqslant C\|U_{1,k}\|_W^{1+\mu}. \tag{5.13}$$

From (4.20) it follows in a similar fashion that

$$\begin{aligned}
&\eta_2(S_{2,k+1}(t)) - \Phi_{k+1}(q_{k+1}\mathrm{e}^{\mathrm{i}t})\\
&\quad= U_{2,k}(t)\int_0^1 (\dot\eta_2(S_{2,k}(t) + \tau U_{2,k}(t)) - \dot\eta_2(S_{2,k}(t)))\mathrm{d}\tau\\
&\qquad+ \mathrm{i}q_{k+1}\gamma(q_{k+1})\exp(w_2(t) + \mathrm{i}\theta_2(S_{2,k}(t))).
\end{aligned}$$

The same argument as before yields

$$\|\eta_2(S_{2,k+1}(t)) - \Phi_{k+1}(q_{k+1}\mathrm{e}^{\mathrm{i}t})\|_W \leqslant C\|U_{2,k}\|_W^{1+\mu} + C\,|\gamma(q_{k+1})|. \tag{5.14}$$

The number q_{k+1} is determined by a quasi-Newton method using $\tilde\gamma'(q_k)$ instead of $\gamma'(q_k)$. In the case $q_{k+1} \geqslant q_k$ the remaining γ is

$$\gamma(q_{k+1}) = \int_{q_k}^{q_{k+1}} (\gamma'(\tau) - \tilde\gamma'(q_k))\mathrm{d}\tau. \tag{5.15}$$

Both γ' and $\tilde\gamma'$ are functions of p and $S_{\nu,k}$. We express this dependence for the moment by writing $\gamma'(p, S_{1,k}, S_{2,k})$. A detailed analysis shows that the dependence is Lipschitz-continuous in p and Hölder-continuous in the $S_{\nu,k}$, i.e.

$$\begin{aligned}
&|\gamma'(p, S_{1,k}, S_{2,k}) - \gamma'(\tilde p, \tilde S_{1,k}, \tilde S_{2,k})|\\
&\quad\leqslant C(|p - \tilde p| + \|S_{1,k} - \tilde S_{1,k}\|_W^\mu + \|S_{2,k} - \tilde S_{2,k}\|_W^\mu).
\end{aligned} \tag{5.16}$$

The same estimate holds for $\tilde\gamma'$. Therefore we obtain from (5.4), (5.6) and (5.7) that for $q_k \leqslant \tau \leqslant q_{k+1}$,

$$|\gamma'(\tau) - \tilde\gamma'(q_k)| \leqslant \epsilon_0 + C\delta_1^\mu,$$

and from (5.15) and (5.11) that

$$|\gamma(q_{k+1})| \leqslant C\epsilon_k(\epsilon_0 + \delta_1^\mu). \tag{5.17}$$

Thus we find from (5.13) and (5.14), (5.12) and (5.17) that (5.8) is also true for $k+1$ with

$$\epsilon_{k+1} \leqslant C\epsilon_k(\epsilon_k^\mu + \epsilon_0 + \delta_1^\mu). \tag{5.18}$$

If $C(\epsilon_k^\mu + \epsilon_0 + \delta_1^\mu) \leqslant c_0 < 1$, then $\epsilon_{k+1} \leqslant c_0\epsilon_k$ decreases geometrically. Therefore in view of (5.12) the series $U_{\nu,0} + U_{\nu,1} + \cdots$ converges in W if $C(\epsilon_0^\mu + \epsilon_0 + \delta_1^\mu) \leqslant c_0 < 1$. This can be achieved by choosing ϵ_0 and δ_1 sufficiently small. It turns out that one can achieve $\delta_1 \leqslant C\epsilon_0$. Therefore the additional hypothesis (5.6) can also be satisfied, by choosing ϵ_0 small. It follows from (5.11) that the sequence q_k also converges, and that (5.7) is true for small ϵ_0. This completes the proof of convergence. It follows furthermore that

$$\|S_{\nu,k} - S_\nu\|_W \leqslant C\epsilon_k, \qquad |q_{k+1} - q| < C\epsilon_k. \tag{5.19}$$

We have shown in Section 4 that $\gamma'(q, S_1, S_2) = \tilde{\gamma}'(q, S_1, S_2)$. In view of (5.19) and the Hölder continuity (5.16) of γ' and $\tilde{\gamma}'$, the integrand in (5.15) can be estimated by $C\epsilon_k^\mu$. Hence $|\gamma(q_{k+1})| \leqslant C\epsilon_k^{1+\mu}$, and it follows from (5.13), (5.14) and (5.12) that one can achieve $\epsilon_{k+1} \leqslant C\epsilon_k^{1+\mu}$, which proves the statement on the order of the convergence. □

If one uses (4.21) instead of (4.39) for the determination of q_{k+1}, then the hypothesis (5.4) is superfluous. For applications it is quite useful that this theorem does not require the knowledge of the conformal mapping Φ. But with the special choice $\Phi_0 = \Phi$ one obtains the following local convergence theorem:

Theorem 2. *If the functions η_ν have Hölder continuous second derivatives $\ddot{\eta}_\nu$, then there exists $\varepsilon > 0$ such that starting with any number q_0 and functions $S_{\nu,0}$ with the properties*

$$|q - q_0| \leqslant \epsilon, \tag{5.20}$$

$$S_{1,0}(0) = 0, \tag{5.21}$$

$$S_{\nu,0}(t) - t \in W_r, \tag{5.22}$$

$$\|S_{\nu,0} - S_\nu\|_W \leqslant \epsilon, \tag{5.23}$$

the iteration converges, i.e.

$$\|S_{\nu,k} - S_\nu\|_W \to 0 \quad \text{and} \quad q_k \to q.$$

Proof. We prepare for the application of Theorem 1 and choose $\delta := \frac{1}{2}\min(q, 1-q)$. It follows from a theorem of Warschawski [6], that the functions $S_\nu(t) - t$ are twice differentiable, hence $S_\nu' \in L^2$. We assume $\epsilon \leqslant 1$ and put

$$M = 1 + \max(\|S_1'\|_2, \|S_2'\|_2).$$

Bu virtue of Lemma 5 one can construct from the boundary values

$$\phi_1(t) := \operatorname{Re} \Phi(e^{it}), \qquad \phi_2(t) := \operatorname{Re} \Phi(q\, e^{it})$$

a family of functions Ψ_p for $0 < p < 1$ which are analytic in the annulus R_p and continuous in $\overline{R}_p$ with boundary values

$$\Psi_p(e^{it}) = \phi_1(t) + i\, K_{p,0}(\phi_1, \phi_2) + ic,$$

$$\Psi_p(p\, e^{it}) = \phi_2(t) - i\, K_{p,0}(\phi_2, \phi_1) + ic.$$

If the constant c is adjusted so that $\Psi_q = \Phi$, then in view of (2.13)

$$\|\Psi_p(e^{it}) - \Phi(e^{it})\|_W \leqslant C|p-q|,$$
$$\|\Psi_p(p\,e^{it}) - \Phi(q\,e^{it})\|_W \leqslant C|p-q|$$

as long as p is bounded away from 0 and 1. For the application of Theorem 1 we choose $\Phi_0 := \Psi_{q_0}$ and obtain the estimates

$$\begin{aligned}
&\|\eta_1(S_{1,0}(t)) - \Phi_0(e^{it})\|_W \\
&\quad \leqslant \|\eta_1(S_{1,0}(t)) - \eta_1(S_1(t))\|_W + \|\Phi(e^{it}) - \Phi_0(e^{it})\|_W \\
&\quad \leqslant C\|S_{1,0} - S_1\|_W + C|q - q_0| \leqslant C\epsilon, \\
&\|\eta_2(S_{2,0}(t)) - \Phi_0(q_0 e^{it})\|_W \\
&\quad \leqslant \|\eta_2(S_{2,0}(t)) - \eta_2(S_2(t))\|_W + \|\Phi(q\,e^{it}) - \Phi_0(q_0 e^{it})\|_W \\
&\quad \leqslant C\|S_{2,0} - S_2\|_W + C|q - q_0| \leqslant C\epsilon.
\end{aligned}$$

Therefore by choosing ϵ sufficiently small one can satisfy (5.5). The left-hand side of (5.4) is a function which depends in a continuous way on $q_0 \in \mathbb{R}$ and $S_{\nu,0}(t) - t \in W_r$. We have shown at the end of Section 4 that this function vanishes for (q, S_1, S_2). Hence ϵ can be chosen so small that (5.4) is satisfied. Application of Theorem 1 now completes the proof. □

6. Numerical implementation

For the numerical approximation of the operator $K_{q,\lambda}$ one can use the same principle which is known from Wittich's method for numerical conjugation on the unit circle. Let N be an even number $N = 2n$, and divide the interval $[0, 2\pi]$ by N equidistant grid points

$$t_\mu := (\mu - 1)2\pi/N, \quad \mu = 1, 2, \ldots, N. \tag{6.1}$$

To each function u which is defined at least at these grid points assign the trigonometric polynomial

$$T_N u(t) := a_0 + \sum_{l=1}^{n} (a_l \cos lt + b_l \sin lt) \tag{6.2}$$

with $T_N u(t_\mu) = u(t_\mu)$ for $\mu = 1, \ldots, N$. This interpolating polynomial is unique if the normalization $b_n = 0$ is imposed. The operator $K_{p,\lambda}$ is then approximated by

$$K_{p,\lambda,N}(\phi_1, \phi_2) := K_{p,\lambda}(T_N\phi_1, T_N\phi_2) \tag{6.3}$$

for $\phi_1, \phi_2 \in W$. Hence $K_{p,\lambda,N}$ is again a trigonometric polynomial of degree n. The trigonometric interpolation as well as the evaluation of $K_{p,\lambda,N}$ at the grid points t_μ can be performed very effectively using the fast Fourier transform. The lth coefficients of $K_{p,\lambda,N}(\phi_1, \phi_2)$ are computed from the lth coefficients of the ϕ_ν by (2.3). This formula is very convenient if one uses an FFT program which yields the coefficients in complex form $a_l - ib_l$. But it can also easily be rewritten in real form. The numerical iteration scheme is then as follows:

Assume that a number q and the values of functions S_1 and S_2 at the grid points t_μ are given.

Then in each iterative step for $\nu = 1, 2$ and $\mu = 1, \ldots, N$, perform the following steps.

Step 1. Calculate

$$v_\nu(t_\mu) := t_\mu - \theta_\nu(S_\nu(t_\mu))$$

and

$$f_\nu(t_\mu) := \operatorname{Im}\big(\exp(-i\theta_\nu(S_\nu(t_\mu)))\eta_\nu(S_\nu(t_\mu))\big).$$

Step 2. Determine the coefficients $a_{\nu,0}, a_{\nu,1}, \ldots, a_{\nu,n}$ and $b_{\nu,1}, \ldots, b_{\nu,n-1}$ of the interpolation polynomials $T_N v_\nu$.

Step 3. Set $\lambda := a_{2,0} - a_{1,0}$.

Step 4. Using the coefficients $a_{\nu,\mu}$ and $b_{\nu,\mu}$ from Step 2, evaluate

$$w_1(t_\mu) := K_{q,0,N}(v_1, v_2)(t_\mu), \qquad w_2(t_\mu) := -K_{q,0,N}(v_2, v_1)(t_\mu),$$

Step 5. Determine

$$g_1(t_\mu) := f_1(t_\mu)\exp(-w_1(t_\mu)), \qquad g_2(t_\mu) := \frac{1}{q} f_2(t_\mu)\exp(-w_2(t_\mu)).$$

Step 6. Calculate the coefficients $a^*_{\nu,0}, a^*_{\nu,1}, \ldots, a^*_{\nu,n}$ and $b^*_{\nu,1}, \ldots, b^*_{\nu,n-1}$ of the interpolation polynomials $T_N g_\nu$.

Step 7. Using the coefficients $a^*_{\nu,\mu}$ and $b^*_{\nu,\mu}$ from Step 6, evaluate the numbers

$$\begin{aligned} &h_1(0) := K_{q,\lambda,N}(g_1, g_2)(0), \\ &\alpha := -\operatorname{Re}(\exp(-w_1(0) - i\theta_1(0))\eta_1(0)) - h_1(0), \\ &\gamma := a^*_{1,0}\cos\lambda - a^*_{2,0} - \alpha\sin\lambda. \end{aligned} \tag{6.4}$$

Step 8. Replace the value $q = q_{\text{old}}$ by the new value $q := q_{\text{new}}$ defined by

$$q_{\text{new}} := q_{\text{old}}\left(1 + \frac{\gamma \cdot 2\pi}{L(\Gamma_1)N} \sum_{\mu=1}^{N} \exp(w_1(t_\mu))\right).$$

Step 9. With this new value of q perform the Steps 4, 5 and 6 again.

Step 10. Using the new values of the coefficients $a^*_{\nu,\mu}$ and $b^*_{\nu,\mu}$ evaluate

$$h_1(t_\mu) := K_{q,\lambda,N}(g_1, g_2)(t_\mu), \qquad h_2(t_\mu) := -K_{q,\lambda,N}(g_2, g_1)(t_\mu).$$

Step 11. Recalculate α from (6.4) using the new values of $w_1(0)$ and $h_1(0)$.

Step 12. Determine the shifts

$$U_1(t_\mu) := -\frac{(h_1(t_\mu) + \alpha)\exp(w_1(t_\mu))}{|\dot\eta_1(S_1(t_\mu))|} - \operatorname{Re}\left(\frac{\eta_1(S_1(t_\mu))}{\dot\eta_1(S_1(t_\mu))}\right),$$

$$U_2(t_\mu) := -\frac{(h_2(t_\mu) + \alpha\cos\lambda + a^*_{1,0}\sin\lambda)q\exp(w_2(t_\mu))}{|\dot\eta_2(S_2(t_\mu))|} - \operatorname{Re}\left(\frac{\eta_2(S_2(t_\mu))}{\dot\eta_2(S_2(t_\mu))}\right)$$

and replace the values of the functions $S_\nu = S_{\nu,\text{old}}$ at the grid points by the new values $S_\nu := S_{\nu,\text{new}}$ defined by

$$S_{\nu,\text{new}}(t_\mu) := S_{\nu,\text{old}}(t_\mu) + U_\nu(t_\mu).$$

Each iteration requires two trigonometric interpolations in each of the Steps 2, 6 and 9 and two evaluations of trigonometric polynomials at all grid points in the Steps 4, 9 and 10. Therefore each iterative step requires 12 real FFTs of length N. For large N these FFTs dominate the cost of the calculation.

7. Numerical experiments

If the region G is bounded by two circles, then the conformal mapping Φ is simply a linear mapping $\Phi(z) = (az+b)/(cz+d)$. The parameters a, b, c, d can easily by determined from the radii and the coordinates of the centers of the circles. Therefore these regions are very suitable for test calculations.

We have done calculations for regions G with the unit circle as outer boundary. The inner boundary is a circle with centre at z_0 and radius r. The iteration was in each case started with $q_0 = r$, $S_{1,0}(t) = t$ and $S_{2,0}(t) = 0.2 + t$. The calculations where done on the Amdahl 470 of the IPP in Garching. The machine accuracy is 2.2×10^{-16}.

In the first table we show the values of some quantities in each iteration for a calculation with $N = 64$ grid points for the example $z_0 = 0.3$, $r = 0.6$. The quadratic convergence is clearly visible. It stops as soon as the numerical error becomes comparable to the error of the iteration. This behavior is quite similar to that of the method for simply connected regions.

It is very tempting to omit Step 9, namely the recalculation of the functions w_ν and g_ν for the new value of q, and to use the values of w_ν and g_ν already determined for the old q in the

Table 1
Evolution of some quantities during the iteration with the complete scheme (first line) and with a modified scheme, in which Step 9 is omitted (second line)

k	$\lVert U_{1,k}\rVert_\infty$	$\lVert U_{2,k}\rVert_\infty$	λ	γ	$q_k - q$
0	1.15	1.06	−0.2	0.55 E−1	−0.11
	1.06	0.98	−0.2	0.56 E−1	−0.1
1	0.89	0.83	0.17 E−1	0.91 E−1	−0.78 E−1
	0.79	0.69	0.16 E−1	0.89 E−1	−0.78 E−1
2	0.18	0.20	−0.14 E−1	0.82 E−2	−0.77 E−2
	0.17	0.13	−0.68 E−3	0.10 E−1	−0.11 E−1
3	0.30 E−1	0.26 E−1	0.17 E−3	−0.30 E−3	0.30 E−3
	0.26 E−1	0.11 E−1	0.67 E−3	0.14 E−2	−0.14 E−2
4	0.35 E−3	0.41 E−4	−0.20 E−4	−0.35 E−6	0.34 E−6
	0.69 E−3	0.10 E−2	0.14 E−4	0.19 E−4	−0.19 E−4
5	0.57 E−7	0.25 E−7	−0.10 E−8	−0.12 E−8	−0.12 E−8
	0.94 E−6	0.13 E−4	0.19 E−7	−0.91 E−7	0.90 E−7
6	0.89 E−10	0.30 E−10	0.0	0.22 E−15	−0.14 E−15
	0.11 E−9	0.63 E−7	0.35 E−12	−0.28 E−10	0.28 E−10
7	0.67 E−10	0.23 E−10	−0.22 E−15	−0.22 E−15	0.14 E−16
	0.48 E−10	0.31 E−10	0.0	−0.44 E−15	0.29 E−15

Table 2
Numerical errors at the outer boundary (E_1), the inner boundary (E_2), and in q for calculations with the method of Theodorsen–Garrick (first line) and with our method (second line) after the kth iteration

z_0	r	k	E_1	E_2	$q_k - q$
0.05	0.9	24	0.24 E−14	0.22 E−14	0.69 E−16
		7	0.10 E−13	0.10 E−13	0.42 E−16
0.1	0.2	54	0.99 E−15	0.53 E−11	0.15 E−15
		7	0.12 E−15	0.94 E−15	0.97 E−16
0.2	0.5	39	0.21 E−14	0.12 E−10	0.26 E−15
		7	0.21 E−14	0.22 E−14	0.15 E−15
0.2	0.7	39	0.21 E−12	0.36 E−9	0.27 E−16
		7	0.39 E−13	0.13 E−13	0.49 E−15
0.3	0.5	79	0.92 E−11	0.24 E−5	0.11 E−12
		7	0.12 E−13	0.56 E−14	0.28 E−15
0.3	0.6	54	0.21 E−9	0.31 E−5	0.57 E−13
		7	0.35 E−10	0.56 E−11	0.14 E−16

remaining Steps 10–12. This would save two FFTs in the repetition of the Steps 4 and 6. As a result one iteration needs only 8 FFTs. We do not have a proof of convergence for this modified method, but in test calculations it has worked quite well. But the convergence seems to be somewhat slower. This is shown by the comparison in Table 1,

For some examples we have also done calculations with the method of Theodorsen–Garrick as described in [3, p. 194 ff.] This method can be implemented very effectively if the FFT is used [4]. In Table 2 we list separately the final errors on both boundary components

$$E_\nu := \max_\mu |\eta_\nu(S_{\nu,k}(t_\mu)) - \eta_\nu(S_\nu(t_\mu))| \tag{7.1}$$

for some calculations with $N = 64$ grid points as well as the errors in the number q. Here we choose k so large that the iteration becomes stationary up to the accuracy of the machine. For comparison we list the corresponding errors after the 7th iteration ($k = 7$ in (7.1)) for calculations with our method.

Only for the first example are the results of the Theodorsen–Garrick calculation more accurate. For the examples with larger z_0 our results have smaller errors. The difference is most pronounced at the inner edge. Using the rule of thumb that one iteration with our method costs as much computer time as three iterations with the method of Theodorsen–Garrick, we see by comparison of the corresponding numbers k in Table 2 that for most examples our method is faster.

It is plausible that error estimates analogous to those of Section 8 of [8] are valid also for doubly connected regions. Therefore we expect an error estimate of the order $e^{-\tau N/2}$ with some $\tau > 0$ in the calculations for the aforementioned region. If z_0 is real, then the conformal mapping is given by $\Phi(z) = (z + a)/(1 + az)$. This function is analytic in the circle $|z| < R := 1/a$. In Fig. 1 we show the numerical error after the 7th iteration for four series of test calculations which were started with $q_0 = r$, $S_{1,0}(t) = t$, $S_{2,0}(t) = 0.1 + t$. It turns out that the errors at the outer and

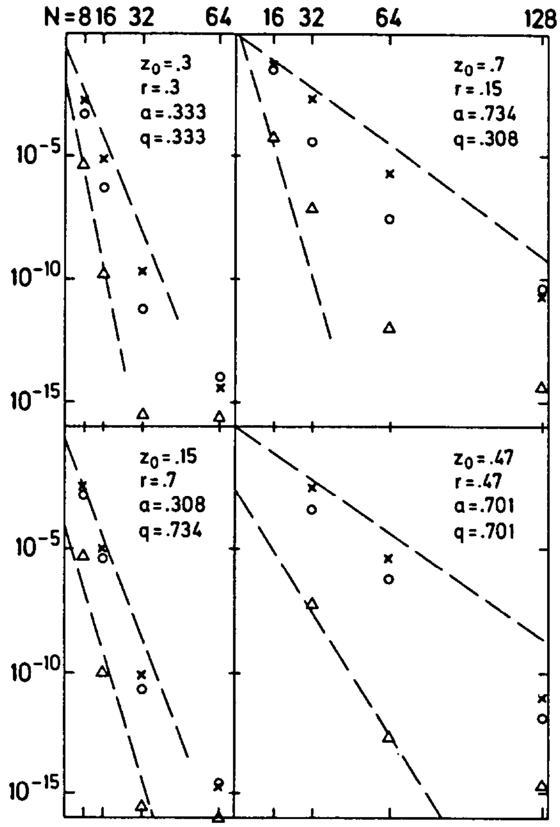

Fig. 1. Numerical errors at the outer edge $E_1(\times)$, at the inner edge $E_2(\circ)$, and in the number $q(\triangle)$ for four series of test calculations with varying number of grid points N, and approximation by $Ca^{N/2}$ and $C(qa)^{N/2}$ (dashed lines).

the inner edge and the error in q are rather well represented by an exponential law $e^{-\tau N/2}$, but with different τ for the different types of errors. These results suggest further that τ satisfies $e^{-\tau} \leqslant a$.

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